

The minimum size of graphs with given rainbow index

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Abstract

The concept of k -rainbow index $rx_k(G)$ of a connected graph G , introduced by Chartrand, Okamoto and Zhang, is a natural generalization of the rainbow connection number. Let $t(n, k, \ell)$ denote the minimum size of a connected graph G of order n with $rx_k(G) \leq \ell$, where $2 \leq \ell \leq n-1$ and $2 \leq k \leq n$. In this paper, we obtain some exact values and some upper bounds for $t(n, k, \ell)$.

Keywords: edge coloring, rainbow connection, rainbow index, rainbow S -tree

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1 Introduction

All graphs considered in this paper are finite, undirected and simple. We follow the notation and terminology of Bondy and Murty [1], unless otherwise stated.

An edge-colored graph G is *rainbow connected* if every two vertices are connected by a path satisfying no two edges on the path have the same color. The minimum number of colors required to make a graph G rainbow connected is called *the rainbow connection number*, denoted $rc(G)$. The notion of rainbow connection in graphs was introduced by Chartrand et al. [3].

For an edge-colored nontrivial connected graph G of order n , A tree T in G is called a *rainbow tree* if no two edges of T are colored the same. For $S \subseteq V(G)$ and $|S| \geq 2$, a *rainbow S -tree* is a rainbow tree T such that $S \subseteq V(T)$. For a fixed integer k with $2 \leq k \leq n$, an edge-coloring c of G is called a *k -rainbow coloring* if for every k -subset S of $V(G)$ there exists a rainbow S -tree. The minimum number of colors that are needed in a k -rainbow coloring of G is called the *k -rainbow index* of G , denoted $rx_k(G)$. Clearly, when $k = 2$, $rx_2(G)$ is the rainbow connection number $rc(G)$ of G . Note that k -rainbow index, defined by Chartrand et al.[4], is a generalization of rainbow connection number. The study about rainbow connection has been extensively researched, we refer to [2, 7, 8, 9, 11] for example.

In this paper, motivated by a recent paper of Schiermeyer [12] on the minimum size of rainbow k -connected graphs, where a graph G is called *rainbow k -connected* if there is an edge colouring of G with k colours such that G is rainbow connected, we study the minimum size of a graph G such that G has a k -rainbow coloring using a fixed number of colors. To be more specific, let $t(n, k, \ell)$ be the minimum size of a connected graph G of order n with $rx_k(G) \leq \ell$, where $2 \leq \ell \leq n-1$ and $2 \leq k \leq n$. Observe that

$$t(n, k, 1) \geq t(n, k, 2) \geq \dots \geq t(n, k, n-1).$$

Our main objective is to give some exact values and some upper bounds for $t(n, k, \ell)$ when k and ℓ take specific values.

2 Main Results

In this section, we mainly concern about some exact values and some upper bounds for $t(n, k, \ell)$, when $k = 3$.

Proposition 2.1 *Let $n \geq 3$ be a positive integer. Then*

(1)

$$t(n, 3, 2) = \begin{cases} 2, & \text{if } n = 3; \\ 4, & \text{if } n = 4; \\ \binom{5}{2}, & \text{if } n = 5. \end{cases}$$

Furthermore, when $n \geq 6$, there does not exist a connected graph G such that $rx_3(G) \leq 2$.

(2)

$$t(n, 3, 3) = \begin{cases} 2, & \text{if } n = 3; \\ 3, & \text{if } n = 4; \\ 5, & \text{if } n = 5. \end{cases}$$

Furthermore, when $n \geq 6$,

$$t(n, 3, 3) \leq \begin{cases} \frac{n^2}{4}, & \text{if } n \text{ is even;} \\ \frac{(n+3)(n-1)}{4}, & \text{if } n \text{ is odd.} \end{cases}$$

To prove Proposition 2.1, we need the following results.

Theorem 2.2 ([4, Proposition 1.3]) *Let T be a tree of order $n \geq 3$. For each integer k with $3 \leq k \leq n$,*

$$rx_k(T) = n - 1.$$

Theorem 2.3 ([4, Theorem 2.1]) *For integers k and n with $3 \leq k \leq n$,*

$$rx_k(C_n) = \begin{cases} n - 2, & \text{if } k = 3 \text{ and } n \geq 4; \\ n - 1, & \text{if } k = n = 3 \text{ or } 4 \leq k \leq n. \end{cases}$$

Theorem 2.4 ([4, Theorem 2.2 and Theorem 2.3]) *If G is a unicyclic graph of order $n \geq 3$ and girth $g \geq 3$ that is not a cycle, then*

$$rx_3(G) = \begin{cases} n - 2, & g \geq 4; \\ n - 1, & g = 3; \end{cases}$$

where a unicyclic graph means a connected graph containing exactly one cycle.

Theorem 2.5 ([5, Theorem 3]) *Let G be a connected graph of order n . Then $rx_3(G) = 2$ if and only if $G = K_5$ or G is a 2-connected graph of order 4 or G is of order 3.*

Theorem 2.6 ([5, Theorem 5]) *For each integer r with $r \geq 3$, $rx_3(K_{r;r}) = 3$.*

Using Theorem 2.2, Theorem 2.3, Theorem 2.4, Theorem 2.5 and Theorem 2.6, it is easy to prove Proposition 2.1, we sketch it as follows.

Proof of Proposition 2.1. (1) The result follows by Theorem 2.5.

(2) It is clearly true when $n = 3$ or $n = 4$. Let G be a graph with 5 vertices and $rx_3(G) \leq 3$. If G is a tree, then from Theorem 2.2, $rx_3(G) = 4$, a contradiction. Thus G must contain a cycle, from Theorem 2.3 and Theorem 2.4, it is easy to deduce that the minimum size of G is 5.

When $n \geq 6$, if n is even, then from Theorem 2.6, we have $rx_3(K_{\frac{n}{2}, \frac{n}{2}}) = 3$. Therefore, it is clear that $t(n, 3, 3) \leq \frac{n^2}{4}$.

If n is odd, let $U = \{u_1, u_2, \dots, u_{\frac{n-1}{2}}\}$, $V = \{v\}$ and $W = \{w_1, w_2, \dots, w_{\frac{n-1}{2}}\}$. Then the join $G[U, W] \vee v$ of a complete bipartite graph $G[U, W]$ and the isolate vertex v is denoted by H with $V(H) = U \cup V \cup W$ and $E(H) = \{(x, v) \mid x \in U \cup W\} \cup \{(x, y) \mid x \in U \text{ and } y \in W\}$. Now let us prove that

$$rx_3(H) = 3.$$

Similar to the proof of Theorem 2.6, define a coloring $c: E(H) \rightarrow \{1, 2, 3\}$ as follows:

$$c(u_i w_j) = \begin{cases} 1, & 1 \leq i = j \leq \frac{n-1}{2}; \\ 2, & 1 \leq i < j \leq \frac{n-1}{2}; \\ 3, & 1 \leq j < i \leq \frac{n-1}{2}; \end{cases} \quad (2.1)$$

and

$$c(xv) = \begin{cases} 2, & \text{if } x \in U; \\ 3, & \text{if } x \in W. \end{cases} \quad (2.2)$$

Now, we show that c is a 3-rainbow coloring of K_H . Let $S \subseteq V(K_H)$ with $|S| = 3$. Then we consider the following two cases.

Case 1. $S \subseteq U \cup W$.

Under this case, since $G[U, W]$ is $K_{\frac{n-1}{2}, \frac{n-1}{2}}$ and $rx_3(K_{\frac{n-1}{2}, \frac{n-1}{2}}) = 3$, then it is routine to verify that there is a rainbow S -tree, see [5].

Case 2. $v \in S$.

Let $S = \{x, y, v\}$. Then if $x \in U$ and $y \in W$, $T = \{vx, vy\}$ is a rainbow S -tree. If $x, y \in U$ with $x = u_i$ and $y = u_j$, then $T = \{vu_i, vw_j, u_j w_j\}$ is a rainbow S -tree. If $x, y \in W$ with $x = w_i$ and $y = w_j$, then $T = \{vw_i, vu_j, u_j w_j\}$ is a rainbow S -tree.

Therefore

$$rx_3(H) \leq 3.$$

Moreover, it is clearly that $rx_3(H) \geq 3$. Thus

$$rx_3(H) = 3.$$

Hence

$$t(n, 3, 3) \leq \frac{(n-1)^2}{4} + (n-1) = \frac{(n+3)(n-1)}{4}.$$

■

Theorem 2.7 *Let $n \geq 3$ be a positive integer. Then*

$$t(n, 3, 4) \leq \binom{n}{2} - n + 1.$$

Proof. Let G be a graph such that \overline{G} is a union of a cycle of order $n - 1$ and an isolated vertex. Let w be the isolated vertex in \overline{G} . Then $d_G(w) = n - 1$. Set

$$V(G) \setminus w = \{v_1, v_2, \dots, v_{n-1}\}.$$

Let $n - 1 = 3r + t$ where $0 \leq t \leq 2$. For $0 \leq j \leq r, 1 \leq i \leq n - 1$, set

$$X_1 = \{v_i \mid i = 3j + 1\},$$

$$X_2 = \{v_i \mid i = 3j + 2\}$$

and

$$X_3 = \{v_i \mid i = 3j + 3\}.$$

Then $G[X_1]$ is a clique or a graph obtained from a clique of order $|G[X_1]|$ by deleting one edge, both $G[X_2]$ and $G[X_3]$ are cliques.

In order to show $rx_3(G) \leq 4$, we provide an edge-coloring $c : E(G) \longrightarrow \{1, 2, 3, 4\}$ defined by

$$c(e) = \begin{cases} 1, & \text{if } e \in E_G[w, X_1] \cup E(G[X_3]); \\ 2, & \text{if } e \in E_G[w, X_2] \cup E(G[X_1]); \\ 3, & \text{if } e \in E_G[w, X_3] \cup E(G[X_2]); \\ 4, & \text{if } e \in E_G[X_1, X_2] \cup E_G[X_1, X_3] \cup E_G[X_2, X_3]. \end{cases}$$

Clearly, $c(wv_{n-1}) = 1$ if and only if $n - 1 = 3r + 1$ for some positive integer r . It suffices to show that there exists a rainbow S -tree for any $S \subseteq V(G)$ with $|S| = 3$.

Case 1: $w \in S$.

Without loss of generality, let

$$S = \{w, v_i, v_j\} \ (i < j).$$

If $c(wv_i) \neq c(wv_j)$, then the tree induced by the edge set $\{wv_i, wv_j\}$ is a rainbow S -tree, as desired. So we assume

$$c(wv_i) = c(wv_j).$$

If

$$c(wv_i) = c(wv_j) = 1,$$

then the tree induced by the edge set

$$\{wv_{i+2}, v_i v_{i+2}, wv_j\}$$

is a rainbow S -tree with colors $\{1, 3, 4\}$. When

$$c(wv_i) = c(wv_j) = 2$$

or

$$c(wv_i) = c(wv_j) = 3,$$

it is easy to verify that the tree induced by the edge set

$$\{wv_{i+2}, v_i v_{i+2}, wv_j\}$$

is a rainbow S -tree, as desired.

Case 2: $w \notin S$.

Without loss of generality, let $S = \{v_i, v_j, v_k\}$ ($i < j < k$). Firstly, if

$$c(wv_i) \neq c(wv_j) \neq c(wv_k),$$

then clearly the tree induced by the edge set

$$\{wv_i, wv_j, wv_k\}$$

is the S -tree with colors $\{1, 2, 3\}$.

Secondly, we consider the case

$$c(wv_i) = c(wv_j) = c(wv_k).$$

Suppose

$$c(wv_i) = c(wv_j) = c(wv_k) = 1,$$

then

$$v_i, v_j, v_k \in X_1 \text{ and } v_j \text{ is adjacent to } v_k.$$

Since

$$c(v_i v_{i+2}) = 4, c(v_j v_k) = 2, c(wv_j) = 1$$

and

$$c(wv_{i+2}) = 3.$$

Therefore, the tree induced by the edge set

$$\{wv_{i+2}, v_i v_{i+2}, v_j v_k, wv_j\}$$

is a rainbow S -tree with colors $\{1, 2, 3, 4\}$, as desired. Similarly, when

$$c(wv_i) = c(wv_j) = c(wv_k) = 2$$

or

$$c(wv_i) = c(wv_j) = c(wv_k) = 3.$$

It is routine to verify that the tree induced by the edge set

$$\{wv_{i+2}, v_i v_{i+2}, v_j v_k, wv_j\}$$

is a rainbow S -tree with colors $\{1, 2, 3, 4\}$.

Finally, we consider the case that only two edges in $\{wv_i, wv_j, wv_k\}$ receive the same color under the coloring c . This implies that there exist

$$v_l, v_m \in S = \{v_i, v_j, v_k\}$$

satisfying $v_l, v_m \in X_p$ for some $p \in \{1, 2, 3\}$ and the left element $v_t \in S \setminus \{v_l, v_m\}$ is in X_q with $q \neq p$. Moreover, v_t must be adjacent to v_l or v_m , without loss of generality, set v_t adjacent to v_m . Let

$$x \in \{1, 2, 3\} \setminus \{p, q\} \text{ and } v_y \in X_x.$$

Then

$$c(v_t v_m) = 4, c(wv_t) = q, c(wv_y) = x \text{ and } c(wv_l) = p.$$

Thus the tree induced by the edge set

$$\{v_t v_m, wv_t, wv_y, wv_l\}$$

is a rainbow S -tree with colors $\{1, 2, 3, 4\}$.

From the above arguments, we conclude that $rx_3(G) \leq 4$ and $t(n, 3, 4) \leq \binom{n}{2} - n + 1$. ■

The join $C_n \vee K_1$ of a cycle C_n and a single vertex is referred to as a *wheel* with n *spokes*, denoted W_n , see [6]. The 3-rainbow index of W_n was given by Chen, Li, Yang and Zhao as follows.

Theorem 2.8 ([5, Theorem 7]) *For $n \geq 3$, the 3-rainbow index of the wheel W_n is*

$$rx_3(W_n) = \begin{cases} 2, & n = 3; \\ 3, & 4 \leq n \leq 6; \\ 4, & 7 \leq n \leq 16; \\ 5, & n \geq 17. \end{cases} \quad (2.3)$$

Thus

$$rx_3(W_n) \leq 5.$$

Since $|E(W_n)| = 2n$, the following result is true.

Proposition 2.9 *Let $n \geq 3$ be an integer. Then*

$$t(n, 3, 5) \leq 2n - 2.$$

Theorem 2.10 *Let $n \geq 3$ be an integer. Then*

$$t(n, 3, 6) \leq 2n - 3.$$

Proof. When $3 \leq n \leq 4$, it is true clearly. When $n \geq 5$, let u be an isolated vertex and v be the center of W_{n-2} . G is the graph by adding an edge between u and v , that is,

$$V(G) = V(W_{n-2}) \cup \{u\}$$

and

$$E(G) = E(W_{n-2}) \cup \{uv\}.$$

Since from Theorem 2.8

$$rx_3(W_{n-2}) \leq 5,$$

it is easy to know that

$$rx_3(G) \leq 6.$$

Therefore, for any positive integer $n \geq 3$, we have

$$t(n, 3, 6) \leq 2n - 3,$$

since $|E(G)| = 2n - 3$. ■

Theorem 2.11 *Let n and ℓ be positive integers satisfying $7 \leq \ell \leq \frac{n-1}{2}$. Then*

$$t(n, 3, \ell) \leq n + t + \binom{t}{2} \left(n - 2 - \left\lfloor \frac{n-2}{\ell-3} \right\rfloor (\ell-3) \right) + \binom{t + \left\lfloor \frac{n-2}{\ell-3} \right\rfloor - \left\lceil \frac{n-2}{\ell-3} \right\rceil}{2} \left(\ell + 1 - n + \left\lfloor \frac{n-2}{\ell-3} \right\rfloor (\ell-3) \right),$$

where $t = \left\lceil \frac{n-2}{\ell-3} \right\rceil$.

Proof. Let

$$t = \left\lceil \frac{n-2}{\ell-3} \right\rceil.$$

For $1 \leq i \leq t-1$, let

$$Q_i = uv_{i,1}v_{i,2} \dots v_{i,\ell-3}w$$

be a path of length $\ell - 2$. If $\lceil \frac{n-2}{\ell-3} \rceil = \lfloor \frac{n-2}{\ell-3} \rfloor$, then let

$$Q_t = uv_{t,1}v_{t,2} \dots v_{t,\ell-3}w$$

be a path of length $\ell - 2$. Otherwise let

$$Q_t = uv_{t,1}v_{t,2} \dots v_{t,s}w$$

be a path of length $s + 1$, where $s = n - \lfloor \frac{n-2}{\ell-3} \rfloor(\ell - 3) - 2$.

Let H be the union of Q_1, Q_2, \dots, Q_t . Let G be the graph obtained from H by adding the edges in

$$\{uw\} \cup \{v_{i_1,j}v_{i_2,j} \mid 1 \leq i_1 \neq i_2 \leq t, 1 \leq j \leq s\} \cup \{v_{i_1,j}v_{i_2,j} \mid 1 \leq i_1 \neq i_2 \leq t, s+1 \leq j \leq \ell-3\}.$$

Note that for each j ($1 \leq j \leq s$) the graph induced by the vertex set

$$\{v_{i,j} \mid 1 \leq i \leq t\}$$

is a complete graph of order t . For each j ($s+1 \leq j \leq \ell-3$), the graph induced by the vertex set

$$\{v_{i,j} \mid 1 \leq i \leq t-1\}$$

is a complete graph of order $t-1$.

In order to prove $rx_3(G) \leq \ell$, we provide an edge-coloring $c : E(G) \rightarrow \{1, 2, \dots, \ell\}$ as follows: If $\lceil \frac{n-2}{\ell-3} \rceil = \lfloor \frac{n-2}{\ell-3} \rfloor$, $c(e)$ is defined to be

$$c(e) = \begin{cases} 1, & \text{if } e = uv_{i,1} \text{ for } 1 \leq i \leq t; \\ j, & \text{if } e = v_{i,j-1}v_{i,j} \text{ for } 1 \leq i \leq t \text{ and } 2 \leq j \leq \ell-3; \\ \ell-2, & \text{if } e = v_{i,\ell-3}w \text{ for } 1 \leq i \leq t; \\ \ell-1, & \text{if } e = uw; \\ \ell, & \text{others.} \end{cases}$$

Otherwise, $c(e)$ is defined to be

$$c(e) = \begin{cases} 1, & \text{if } e = uv_{i,1}, \text{ for } 1 \leq i \leq t; \\ j, & \text{if } e = v_{i,j-1}v_{i,j} \text{ or } e = v_{t,p-1}v_{i,q}, \text{ for } 1 \leq i \leq t-1, 2 \leq j \leq \ell-3 \text{ and } 2 \leq q \leq s; \\ \ell-2, & \text{if } e = v_{i,\ell-3}w \text{ or } e = v_{t,s}w, \text{ for } 1 \leq i \leq t-1; \\ \ell-1, & \text{if } e = uw; \\ \ell, & \text{others.} \end{cases}$$

To show $rx_3(G) \leq \ell$, it suffices to prove that there exists a rainbow S -tree for any $S \subseteq V(G)$ with $|S| = 3$. Set

$$S = \{x, y, z\}.$$

Case 1. $S \subset V(Q_i)$ for some $1 \leq i \leq t$.

It is easy to know that Q_i is the required rainbow S -tree.

Case 2. Two of vertices in S are on Q_i for some $1 \leq i \leq t$.

Without loss of generality, let $x, y \in V(Q_i)$ and $z \in V(Q_j)$, where $j \neq i$ and z is neither u nor w . Suppose

$$x = v_{i,p}, y = v_{i,q} \text{ and } z = v_{j,r},$$

where $p \leq q$. If $p \leq q \leq r$, then the path

$$P = v_{i,p}v_{i,p+1} \dots v_{i,q-1}v_{i,q}v_{j,q}v_{j,q+1} \dots v_{j,r-1}v_{j,r}$$

is a rainbow S -tree. If $r \leq p \leq q$ or $p \leq r \leq q$, it is similar, so we omit.

Case 3. For any $1 \leq i \leq t$, $S \cap V(Q_i) = 1$.

Suppose

$$x = v_{i,p}, y = v_{j,q} \text{ and } z = v_{k,r},$$

where i, j, k, p, q and r are positive integers satisfying $1 \leq i \neq j \neq k \leq t$ and $p \leq q \leq r$. If

$$\lceil \frac{n-2}{\ell-3} \rceil = \lfloor \frac{n-2}{\ell-3} \rfloor \text{ or } k \neq t,$$

then the path

$$P' = v_{k,r}v_{k,r+1} \dots v_{k,\ell-3}wuv_{i,1}v_{i,2} \dots v_{i,p}v_{j,p}v_{j,p+1} \dots v_{j,q}$$

is a rainbow S -tree. If

$$\lceil \frac{n-2}{\ell-3} \rceil \neq \lfloor \frac{n-2}{\ell-3} \rfloor \text{ and } k = t,$$

then the path

$$P' = v_{k,r}v_{k,r+1} \dots v_{k,s}wuv_{i,1}v_{i,2} \dots v_{i,p}v_{j,p}v_{j,p+1} \dots v_{j,q}$$

is a rainbow S -tree.

Therefore,

$$rx_3(G) \leq \ell.$$

Since

$$|E(G)| = n + t + \binom{t}{2} \left(n - 2 - \lfloor \frac{n-2}{\ell-3} \rfloor (\ell-3) \right) + \left(t + \lfloor \frac{n-2}{\ell-3} \rfloor - \lceil \frac{n-2}{\ell-3} \rceil \right) \left(\ell + 1 - n + \lfloor \frac{n-2}{\ell-3} \rfloor (\ell-3) \right),$$

the theorem is confirmed. \blacksquare

A *rose graph* R_p with p petals (or p -rose graph) is a graph obtained by taking p cycles with just a vertex in common. The common vertex is called the *center* of R_p . If the length of each cycle is exactly q , then this rose graph with p petals is called a (p, q) -rose graph, denoted $R_{p,q}$.

Theorem 2.12 For $\frac{n}{2} \leq \ell \leq n-3$,

$$t(n, 3, \ell) \leq 2n - \ell - 1.$$

Proof. Let G be a graph obtained from a $(n-\ell, 3)$ -rose graph $R_{n-\ell,3}$ and a path $P_{2\ell-n}$ by identifying the center of the rose graph and one endpoint of the path. Clearly,

$$|V(G)| = (2(n-\ell) + 1) + (2\ell - n - 1) = n$$

and

$$|E(G)| = 3(n-\ell) + (2\ell - n - 1) = 2n - \ell - 1.$$

Let w_0 be the center of $R_{n-\ell,3}$, and let $C_i = w_0v_iu_iw_0$ ($1 \leq i \leq n-\ell$) be a cycle of $R_{n-\ell,3}$. Let $P_{2\ell-n} = w_0w_1 \dots w_{2\ell-n-1}$ be the path of order $2\ell-n$. To show that $rx_3(G) \leq \ell$, we provide an edge-coloring $c : E(G) \rightarrow \{1, 2, \dots, \ell\}$ defined by

$$c(e) = \begin{cases} i, & \text{if } e = w_0u_i \text{ or } e = w_0v_i, \text{ for } 1 \leq i \leq n-\ell; \\ n-\ell+i, & \text{if } e = w_{i-1}w_i, \text{ for } 1 \leq i \leq 2\ell-n-1; \\ \ell, & \text{if } e = u_iv_i, \text{ for } 1 \leq i \leq n-\ell; \end{cases}$$

It suffices to show that there exists a rainbow S -tree for any $S \subseteq V(G)$ and $|S| = 3$. Set

$$S = \{x, y, z\}.$$

Case 1 $w_0 \in S$.

Set $x = w_0$. If

$$y, z \in V(C_i) \setminus \{w_0\}$$

for some $1 \leq i \leq n - \ell$, then the graph induced by the edge set $\{xy, yz\}$ is a rainbow S -tree. If

$$y \in V(C_i) \setminus \{w_0\}, z \in V(C_j) \setminus \{w_0\}$$

for some $1 \leq i \neq j \leq n - \ell$, then the graph induced by the edge set $\{xy, xz\}$ is a rainbow S -tree. If

$$y \in V(C_i) \setminus \{w_0\}, z \in V(P_{2\ell-n}) \setminus \{w_0\}$$

for some $1 \leq i \leq n - \ell$, then the path $yxw_1w_2 \dots z$ is a rainbow S -tree. If

$$y, z \in V(P_{2\ell-n}) \setminus \{w_0\},$$

then $P_{2\ell-n}$ is a rainbow S -tree.

Case 2 $S \subset V(P_{2\ell-n}) \setminus \{w_0\}$.

Clearly, $P_{2\ell-n}$ is a rainbow S -tree.

Case 3 $S \subset V(R_{n-\ell,3}) \setminus \{w_0\}$.

If

$$x, y \in V(C_i) \text{ and } z \in V(C_j)$$

for some $1 \leq i \neq j \leq n - \ell$, then the path zw_0xy is a rainbow S -tree. If

$$x \in V(C_i), y \in V(C_j) \text{ and } z \in V(C_k)$$

for some $1 \leq i \neq j \neq k \leq n - \ell$, then the star induced by the edge set $\{w_0x, w_0y, w_0z\}$ is a rainbow S -tree.

Case 4

$$|S \cap (V(R_{n-\ell,3}) \setminus \{w_0\})| + |S \cap (V(P_{2\ell-n,3}) \setminus \{w_0\})| = 3$$

and

$$|S \cap (V(R_{n-\ell,3}) \setminus \{w_0\})| \cdot |S \cap (V(P_{2\ell-n,3}) \setminus \{w_0\})| \neq 0.$$

Similarly, it is routine to verify that there must exist a rainbow S -tree in G .

From the above arguments, we achieve that

$$rx_3(G) \leq \ell$$

and

$$t(n, 3, \ell) \leq 2n - \ell - 1, \text{ for } \frac{n}{2} \leq \ell \leq n - 3.$$

■

Proposition 2.13 *Let $n \geq 4$ be a positive integer. Then*

$$(1) \ t(n, 3, n - 2) = n;$$

$$(2) \ t(n, 3, n - 1) = n - 1.$$

To prove Proposition 2.13, we need the following results.

Theorem 2.14 ([10, Theorem 3]) *Let G be a connected graph of order n . Then $rx_3(G) = n - 1$ if and only if G is a tree or G is a unicyclic graph with girth 3.*

Combining Theorem 2.3 and Theorem 2.14, Proposition 2.13 is clearly true.

We conclude this paper with the following theorem about an upper bound of $t(n, n - 1, n - 2)$.

Theorem 2.15 $t(n, n - 1, n - 2) \leq 2n - 4$.

Proof. Let $G = K_{2, n-2} = G[X, Y]$, where set $X = \{u, w\}$ and $Y = \{v_1, v_2, \dots, v_{n-2}\}$. Then we give an edge coloring c of G as follows:

$$c(e) = \begin{cases} i, & e = uv_i, \text{ for } 1 \leq i \leq n - 2; \\ n - 1 - i, & e = wv_i, \text{ for } 1 \leq i \leq n - 2. \end{cases}$$

We prove that G is rainbow 3-tree-connected under this coloring. Let

$$S \subseteq V(G) \text{ and } |S| = n - 1.$$

If $S = V(G) \setminus \{u\}$, then the tree T induced by the edge set

$$\{wv_1, wv_2, \dots, wv_{n-2}\}$$

is a rainbow S -tree. If $S = V(G) \setminus \{w\}$, there exists a rainbow S -tree similarly. If $S = V(G) \setminus \{v_i\}$, then the tree T induced by the edge set

$$\{uv_1, uv_2, \dots, uv_{i-1}, uv_{i+1}, \dots, uv_{n-2}, wv_{n-i-1}\}$$

is a rainbow S -tree. Therefore we conclude

$$rx_{n-1}(G) \leq n - 2.$$

Since $e(G) = 2n - 4$, it follows that $t(n, n - 1, n - 2) \leq 2n - 4$. ■

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